*Minimal Surfaces in Euclidean Four-Space.**

By Luther Pfahler Eisenhart.

Introduction.

By definition a minimal curve in Euclidean four-space, Σ_4, is one whose coordinates x, y, z, t satisfy the condition

$$dx^2 + dy^2 + dz^2 + dt^2 = 0.$$

Recently we have observed† that such a curve can be defined in terms of a unique parameter, so that its coordinates are expressible in the form

$$\left.\begin{aligned} x &= \phi - u\phi' + \psi', & iy &= \phi - u\phi' - \psi', \\ z &= \psi - u\psi' - \phi', & it &= -\psi + u\psi' - \phi', \end{aligned}\right\} \quad (1)$$

where $i = \sqrt{-1}$.

If we have two such minimal curves in Σ_4, the locus of the mid-points of all joins of points on the two curves is a two-dimensional spread in Σ_4 which, being analogous to a minimal surface in ordinary space as defined by Lie,‡ we call a *minimal surface in* Σ_4. This paper is a study of minimal surfaces in Σ_4 and sets forth many results analogous to the established theorems concerning ordinary minimal surfaces.

As introductory to the subsequent study, § 1 deals with minimal curves in Euclidean four-space defined in the form (1), and in particular the transformations of u, ϕ, ψ corresponding to rotation of the coordinate axes and to reflection of the curve in a hyperplane. The section closes with a discussion of certain plane minimal curves.

In § 2 certain properties of the general two-dimensional spread in four-space are stated, generally without proof, and the general equations are given

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* Read before the American Mathematical Society, September 12, 1911.

† "A Fundamental Parametric Representation of Space Curves," *Annals of Mathematics*, Series II, Vol. XIII (1911), pp. 17–35.

‡ "Beiträge zur Theorie der Minimalflächen," *Mathematische Annalen*, Vol. XIV (1879), p. 339.

for future use. These results are taken from memoirs by Karl Kommerell* and Levi,† and the reader is referred to these papers for the proofs of the theorems and formulas.

A minimal surface may be defined as one for which the first variation of the area-integral vanishes. In § 3 this definition is shown to be equivalent to the one previously used, and the nature of the characteristic is observed. With the introduction in § 4 of the normal parameters and functions for the minimal curves on the surface the formulas assume elegant forms, from which follows a direct discussion of the properties of the surface in the neighborhood of a point. The changes of parameters produced by the rotation of the surfaces and certain results of this investigation are studied in § 5.

Kommerell‡ has studied the surfaces in four-space defined by an equation of the form

$$z + it = f(x + iy),$$

where f is analytic in the domain considered, and he has shown that these surfaces are minimal. They are considered briefly in § 6 from the point of view of this paper.

When the normal functions ϕ, ψ and ϕ_0, ψ_0 are algebraic functions of u and u_0 respectively, the minimal surface is algebraic. The proof of the converse is given in § 7. It is shown in § 8 that with each minimal surface there are associated a group of applicable minimal surfaces which are analogous to the *associated minimal surfaces* of ordinary space. We use the same term for these surfaces and obtain a number of theorems which set forth other relations between the surfaces of such a family.

§ 1. *Minimal Curves. Rotations and Reflections.*

From equations (1) we have

$$\frac{dx + i\,dy}{dz + i\,dt} = -\frac{dz - i\,dt}{dx - i\,dy} = u. \tag{2}$$

When a minimal curve is defined in terms of a general parameter, equations (2) enable one to determine the *normal parameter* u in terms of which the equations of the curve assume the form (1). From its method of derivation this parameter

* "Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen," Thesis, Tübingen, 1897; also, "Riemann'sche Flächen im ebenen Raum von vier Dimensionen," *Mathematische Annalen*, Vol. LX (1905), pp. 548–596.

† "Saggio sulla theoria delle superficie a due dimensioni immerse in un iperspazio," Thesis, Pisa, 1905.

‡ *Loc. cit.*, p. 548.

u is necessarily unique. Moreover, the *normal functions* ϕ and ψ are determined by the quadratures

$$\left.\begin{aligned}\frac{d\phi}{du} &= -\tfrac{1}{2}(z+it),\\ \frac{d\psi}{du} &= \tfrac{1}{2}(x-iy).\end{aligned}\right\}\qquad(2')$$

It is important for us to study the effect upon the form (1) of a rotation of coordinate axes. It is a known fact that the most general rotation in Euclidean four-space may be decomposed into two *simple rotations* each of which leaves a plane two-spread unaltered point for point, and these two invariant planes are perpendicular to one another with only one point in common.* The equations defining the change of coordinates due to a *simple rotation* may be written

$$\left.\begin{aligned}Bx_1 &= (1+f^2-a^2+g^2-b^2+h^2-c^2)x+2(a-bh+cg)y\\ &\quad+(b-cf+ah)z+2(c-ag+bf)t,\\ By_1 &= 2(-a+cg+ah)x+(1+f^2-a^2+b^2-g^2+c^2-h^2)y\\ &\quad+2(h+fg-ab)z+2(-g+hf-ac)t,\\ Bz_1 &= 2(-b-cf+ah)x+2(-h+fg-ah)y\\ &\quad+(1+g^2-f^2+c^2-h^2+a^2-b^2)z+2(f+gh-bc)t,\\ Bt_1 &= 2(-c+bf-ag)x+2(g+fh-ac)y\\ &\quad+2(-f+gh-bc)z+(1+h^2-c^2+a^2-f^2+b^2-g^2)t,\end{aligned}\right\}\qquad(3)$$

where a, b, c, f, g, h, B are constants satisfying the conditions

$$\left.\begin{aligned}af+bg+ch&=0,\\ B&=1+a^2+b^2+c^2+f^2+g^2+h^2.\end{aligned}\right\}\qquad(4)$$

In this rotation the invariant plane is defined by

$$\left.\begin{aligned}(a^2+b^2+c^2)x-(a-bh+cg)y-(b-cf+ah)z&\\ -(c-ag+bf)t&=0,\\ (a-cg-bh)x+(a^2+g^2+h^2)y-(h+fg-ab)z&\\ +(g-hf+ac)t&=0.\end{aligned}\right\}\qquad(5)$$

The simple rotation, whose invariant plane is perpendicular to (5) and has only one point in common with it, is defined by equations similar to (3) in which the constants a', b', c', f', g', h' are given by

$$a'=ef,\quad b'=eg,\quad c'=eh,\quad f'=ea,\quad g'=eb,\quad h'=ec,\qquad(6)$$

where $e=\pm 1$.

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* Cf. Cole, "On Rotations in Space of Four Dimensions," AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), pp. 199–210; also Cayley, "Sur quelques propriétés des déterminants gauches," *Crelle*, Vol. XXXII (1846), pp. 119–123.

The minimal curve given by (1) may be defined by equations of the same form for x_1, y_1, z_1, t_1 in terms of three functions, u_1, ϕ_1, ψ_1. From (1) and (2) it follows that

$$u_1 = \frac{dx_1 + i\,dy_1}{dz_1 + i\,dt_1},$$
$$\phi_1 = \tfrac{1}{2}\left[x_1 + iy_1 - u_1(z_1 + it_1)\right],$$
$$\psi_1 = \tfrac{1}{2}\left[z_1 - it_1 + u_1(x_1 - iy_1)\right].$$

By means of (1) and (3) these may be reduced to

$$u_1 = \frac{[1 - i(a-f)]\,u + [(b-g) + i(h-c)]}{[-(b-g) + i(h-c)]\,u + [1 + i(a-f)]}, \tag{7}$$

and

$$\left.\begin{aligned} \phi_1 &= \frac{[1 - i(a+f)]\,\phi + [(g+b) + i(c+h)]\,\psi}{[-(b-g) + i(h-c)]\,u + [1 + i(a-f)]}, \\ \psi_1 &= \frac{-[(g+b) - i(c+h)]\,\phi + [1 + i(a+f)]\,\psi}{[-(b-g) + i(h-c)]\,u + [1 + i(a-f)]}. \end{aligned}\right\} \tag{8}$$

In order to obtain the equations of a general rotation, that is, one which does not leave a plane invariant, we must combine the foregoing with a similar transformation suggested by (6). We consider the two cases separately.

If we denote by u_2, ϕ_2, ψ_2 the functions arising from u_1, ϕ_1, ψ_1 when the constants have the values (6) with $e=1$, the equations of the transformation are readily obtained from (7) and (8). Applying the two transformations consecutively, we have

$$\left.\begin{aligned} u_2 &= u, \\ \phi_2 &= \frac{2}{B}\left\{\left[1 - \frac{B}{2} - i(a+f)\right]\phi + \left[g + b + i(c+h)\right]\psi\right\}, \\ \psi_2 &= \frac{2}{B}\left\{-\left[(g+b) - i(c+h)\right]\phi + \left[1 - \frac{B}{2} + i(a+f)\right]\psi\right\}. \end{aligned}\right\} \tag{9}$$

Proceeding in like manner with the case $e = -1$ in (6), we obtain

$$\left.\begin{aligned} u_2 &= \frac{\left[1 - \frac{B}{2} - i(a-f)\right]u + \left[b - g + i(h-c)\right]}{-\left[(b-g) - i(h-c)\right]u + \left[1 - \frac{B}{2} + i(a-f)\right]}, \\ \phi_2 &= \frac{B\phi}{U_1}, \qquad \psi_2 = \frac{B\psi}{U_1}, \end{aligned}\right\} \tag{10}$$

where U_1 denotes the denominator of the expression for u_2.

Conversely, it can be shown that the minimal curve whose normal parameter and functions are

$$u_1 = u, \qquad \phi_1 = \frac{a_1\phi + a_2\psi}{A}, \qquad \psi_1 = \frac{a_3\phi + a_4\psi}{A}, \tag{11}$$

where a_1, a_2, a_3, a_4 are constants and

$$A^2 = a_1 a_4 - a_2 a_3, \tag{12}$$

differs from the curve (2) by the rotation defined by

$$\left.\begin{aligned}
x_1 &= \frac{1}{2A}\left[(a_1 + a_4)x + (a_1 - a_4)iy + (a_2 - a_3)z - e(a_2 + a_3)it\right],\\
y_1 &= \frac{1}{2A}\left[(a_4 - a_1)ix + (a_1 + a_4)y - (a_2 + a_3)iz + e(a_3 - a_2)t\right],\\
z_1 &= \frac{1}{2A}\left[(a_3 - a_2)x + (a_2 + a_3)iy + (a_1 + a_4)z + e(a_1 - a_4)it\right],\\
t_1 &= \frac{1}{2A}\left[e(a_2 + a_3)ix + e(a_2 - a_3)y + e(a_4 - a_1)iz + (a_1 + a_4)t\right],
\end{aligned}\right\} \tag{13}$$

where $e = 1$.

In like manner the curve for which

$$u_1 = \frac{a_1 u + a_2}{a_3 u + a_4}, \qquad \phi_1 = \frac{A\phi}{a_3 u + a_4}, \qquad \psi_1 = \frac{A\psi}{a_3 u + a_4} \tag{14}$$

differs from (2) by the rotation defined by (13), where $e = -1$.

It should be remarked that a necessary and sufficient condition that the rotation be real is that a_1, a_4 and a_2, $-a_3$ be pairs of conjugate imaginaries, which condition is satisfied in the case of (9) and of (10).

Having thus considered the change in the normal parameter due to a rotation, we consider next the effect of a reflection in one of the coordinate hyperplanes. For example, the reflection in the plane $t = 0$ of the curve (1) gives a curve whose equations are the same as (2) except the last, which merely differs in sign. If we write the equations of the reflected curve in the same form as (1), but in terms of u_1, ϕ_1, ψ_1, it is readily found that*

$$\left.\begin{aligned}
&u_1 = \frac{\phi''}{\psi''}, \qquad \psi_1 = \frac{\psi'\phi'' - \phi'\psi''}{\psi''},\\
&\phi_1 = \phi - u\phi' + (u\psi' - \psi)\frac{\phi''}{\psi''}.
\end{aligned}\right\} \tag{15}$$

* This result coincides with the transformation from one normal parameter to the other of a twisted curve in three-space. Cf. *Annals of Mathematics*, *loc. cit.*, p. 19.

The reflections in the other coordinate hyperplanes lead to similar results. In a more general way one may find the transformations corresponding to a reflection in any hyperplane whatsoever.

A particular minimal curve which is of interest in the subsequent study is that for which

$$\phi = c\psi, \tag{16}$$

where c denotes a constant. In this case equations (1) reduce to

$$\left.\begin{aligned} x &= c\psi + (1 - cu)\psi', & iy &= c\psi - (1 + cu)\psi', \\ z &= \psi - (u + c)\psi', & it &= -\psi + (u - c)\psi'. \end{aligned}\right\} \tag{17}$$

From these equations we find that the curve lies in the plane which is the intersection of the two isotropic hyperplanes

$$\left.\begin{aligned} c(x - iy) + (z + it) &= 0, \\ x + iy - c(z - it) &= 0. \end{aligned}\right\} \tag{18}$$

For the minimal curve (17) equations (15) reduce to

$$u_1 = c, \qquad \phi_1 = 0, \qquad \psi_1 = 0.$$

Hence, the curve

$$\left.\begin{aligned} x &= c\psi + (1 - cu)\psi', & iy &= c\psi - (1 + cu)\psi', \\ z &= \psi - (u + c)\psi', & it &= \psi - (u - c)\psi', \end{aligned}\right\} \tag{19}$$

can not be defined by equations of the form (1). Conversely, it follows from (2) that a minimal curve whose equations can not be put in the form (1) lies in a plane whose equations are

$$x + iy - c(z + it) = a, \qquad c(x - iy) + (z - it) = b, \tag{20}$$

where a, b, c are constants.

§ 2. *Two-Dimensional Spreads in Four-Space and Their Properties.*

The general two-dimensional spread or surface in four-space may be defined by equations of the form

$$x = \phi_1(u, v), \quad y = \phi_2(u, v), \quad z = \phi_3(u, v), \quad t = \phi_4(u, v), \tag{21}$$

where the functions ϕ_i are analytic within the domain considered.

The tangent at the point (u, v) to the curve on the surface determined by a relation between u and v has equations of the form

$$\left.\begin{aligned} X - x &= \lambda(x_{10}\,du + x_{01}\,dv), & Y - y &= \lambda(y_{10}\,du + y_{01}\,dv), \\ Z - z &= \lambda(z_{10}\,du + z_{01}\,dv), & T - t &= \lambda(t_{10}\,du + t_{01}\,dv), \end{aligned}\right\} \tag{22}$$

where λ denotes a function of u and v and

$$x_{ij} = \frac{\partial^{i+j} x}{\partial u^i\,\partial v^j}.$$

If we consider these equations for all values of du and dv, they define a plane which we call the *tangent plane* at the point. This tangent plane is the plane common to the one-parameter family of tangent hyperplanes whose equation is

$$a(X-x)+b(Y-y)+c(Z-z)+d(T-t)=0, \tag{23}$$

with

$$a x_{10}+b y_{10}+c z_{10}+d t_{10}=0, \qquad a x_{01}+b y_{01}+c z_{01}+d t_{01}=0. \tag{24}$$

The plane which is normal to the tangent plane at (u, v) and has only this point in common with it is called the *normal plane* to the surface at the point. Its equations are

$$\left.\begin{aligned}(X-x)x_{10}+(Y-y)y_{10}+(Z-z)z_{10}+(T-t)t_{10}=0,\\ (X-x)x_{01}+(Y-y)y_{01}+(Z-z)z_{01}+(T-t)t_{01}=0.\end{aligned}\right\} \tag{25}$$

The hyperplane whose equation is a linear combination of these equations is called a *normal hyperplane.* There is a one-parameter family of normal hyperplanes at each point of a surface.

The normal planes at the points (u, v) and $(u+du,\ v+dv)$ meet in the point determined by the planes (25) and

$$\left.\begin{aligned}\Sigma(X-x)(x_{20}\,du+x_{11}\,dv)-\Sigma x_{10}^2\,du-\Sigma x_{10}x_{01}\,dv=0,\\ \Sigma(X-x)(x_{11}\,du+x_{02}\,dv)-\Sigma x_{01}x_{10}\,du-\Sigma x_{01}^2\,dv=0.\end{aligned}\right\} \tag{26}$$

Solving these four equations for $X-x,\ Y-y,\ \ldots,\ T-t,$ we have

$$\left.\begin{aligned}X-x&=\frac{(ED'_x-FD_x)\,du^2+(ED''_x-GD_x)\,du\,dv+(FD''_x-GD'_x)\,dv^2}{e\,du^2+2f\,du\,dv+g\,dv^2},\\ Y-y&=\frac{(ED'_y-FD_y)\,du^2+(ED''_y-GD_y)\,du\,dv+(FD''_y-GD'_y)\,dv^2}{e\,du^2+2f\,du\,dv+g\,dv^2},\\ Z-z&=\ldots\ldots\ldots\ldots\ldots\ldots,\\ T-t&=\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}\right\} \tag{27}$$

where

$$D_x=\begin{vmatrix}y_{10} & z_{10} & t_{10}\\ y_{01} & z_{01} & t_{01}\\ y_{20} & z_{20} & t_{20}\end{vmatrix}, \quad D'_x=\begin{vmatrix}y_{10}, & \cdots\\ y_{01}, & \cdots\\ y_{11}, & \cdots\end{vmatrix}, \quad D''_x=\begin{vmatrix}y_{10}, & \cdots\\ y_{01}, & \cdots\\ y_{02}, & \cdots\end{vmatrix}, \tag{28}$$

and $D_y, \ldots, D''_t$ denote similar functions, and where

$$e=\begin{vmatrix}x_{10}, & \cdots\\ x_{01}, & \cdots\\ x_{20}, & \cdots\\ x_{11}, & \cdots\end{vmatrix}, \quad 2f=\begin{vmatrix}x_{10}, & \cdots\\ x_{01}, & \cdots\\ x_{20}, & \cdots\\ x_{02}, & \cdots\end{vmatrix}, \quad g=\begin{vmatrix}x_{10}, & \cdots\\ x_{01}, & \cdots\\ x_{11}, & \cdots\\ x_{02}, & \cdots\end{vmatrix}. \tag{29}$$

If dv/du be eliminated from equations (27), we find that all the points thus determined in the normal plane at the point (u, v) generate a conic, which Kommerell* has called the "characteristic". In certain ways this conic is the analogue of the Dupin indicatrix of an ordinary surface in the study of the geometry of the surface in the neighborhood of the point.

We have seen that through each point (u, v) of a surface S there passes a single parameter family of hyperplanes normal to S, and that these hyperplanes have in common a unique plane — the normal plane to the surface at the point. Each of these normal hyperplanes meets the surface in a curve whose center of first curvature lies in the normal plane. Kommerell† and Levi‡ have shown that the locus of these centers of curvature is the first pedal curve of the characteristic with respect to the point (u, v) of the surface, that is, a lemniscate. Hence there are four maxima and minima radii of normal curvature. They are called the *principal radii of normal curvature*, and the corresponding directions of the normal hyperplane are called the *principal directions*. The curves on the surface determined by these directions are called the *lines of curvature*.

Kommerell shows§ that the lines of curvature are characterized by the property that the corresponding point of the characteristic lies in the corresponding osculating plane of the curve. By means of this result it may be shown that the equation of the lines of curvature is

$$\begin{aligned}\Sigma\,(D_x\,du^2 + 2\,D'_x\,du\,dv + D''_x\,dv^2)\,[(E D'_x - F D_x)\,du^2 \\ + (E D''_x - G D_x)\,du\,dv + (F D''_x - G D'_x)\,dv^2] = 0,\end{aligned} \tag{30}$$

and that the principal radii of curvature are given by

$$\rho^2 = \frac{(E G - F^2)\,(E\,du^2 + 2\,F\,du\,dv + G\,dv^2)^2}{\Sigma\,(D_x\,du^2 + 2\,D'_x\,du\,dv + D''_x\,dv^2)^2}, \tag{31}$$

when the values dv/du from (30) are substituted in this equation.‖

§ 3. *Definition of Minimal Surfaces.*

As in the case of ordinary surfaces, we put

$$E = \Sigma\,x_{10}^2, \qquad F = \Sigma\,x_{10}\,x_{01}, \qquad G = \Sigma\,x_{01}^2, \tag{32}$$

and

$$H = \sqrt{E G - F^2}, \tag{33}$$

where the summation extends to terms in x, y, z, t. The element of area of the

* *Loc. cit.*, p. 22; p. 560.
† *Loc. cit.*, p. 50; p. 563.
‡ *Loc. cit.*, pp. 67, 68.
§ *Loc. cit.*, p. 37; p. 566.
‖ Cf. Levi, *loc. cit.*, p. 58.

surface is given by $H\,du\,dv$, and the area of a portion of the surface has the value

$$\sigma = \iint H\,du\,dv. \tag{34}$$

By definition a *minimal surface* is one which minimizes the integral (34) for a given contour curve. We proceed to the determination of the conditions to be satisfied by the equations of a surface in order that it be a minimal surface.

The first variation of (34) may be written

$$\begin{aligned}\delta\sigma &= \iint \sum \left(\frac{\partial X}{\partial x_{10}}\delta x_{10} + \frac{\partial H}{\partial x_{01}}\delta x_{01}\right) du\,dv \\ &= -\iint \sum \left[\frac{\partial}{\partial u}\frac{\partial H}{\partial x_{10}} + \frac{\partial}{\partial v}\frac{\partial H}{\partial x_{01}}\right]\delta x\,du\,dv.\end{aligned}$$

In order that $\delta\sigma$ be zero, the expression in the brackets, and similar expressions in y, z, t, must be zero. We note that

$$\frac{\partial H}{\partial x_{10}} = \frac{1}{H}(x_{10}G - x_{01}F), \qquad \frac{\partial H}{\partial x_{01}} = \frac{1}{H}(x_{01}E - x_{10}F).$$

Hence the vanishing of the four parentheses requires that x, y, z, t satisfy the equation

$$G\theta_{20} - 2F\theta_{11} + E\theta_{02} + \theta_{10}\left[\frac{\partial}{\partial v}\left(\frac{F}{H}\right) - \frac{\partial}{\partial u}\left(\frac{G}{H}\right)\right] + \theta_{01}\left[\frac{\partial}{\partial u}\left(\frac{F}{H}\right) - \frac{\partial}{\partial v}\left(\frac{E}{H}\right)\right] = 0.$$

Thus far the parametric curves have been perfectly general. Let us assume that they are the lines of length zero, or minimal lines, so that $E = G = 0$. In this case the above equation reduces to $\theta_{11} = 0$. Since the converse of this result is evidently true, we have the theorem:*

The general minimal surface in four-space is a surface of translation whose generating curves are minimal and conversely.

That is, the general minimal surface is defined by

$$\left.\begin{aligned} x &= f_1(u) + \phi_1(v), & y &= f_2(u) + \phi_2(v), \\ z &= f_3(u) + \phi_3(v), & t &= f_4(u) + \phi_4(v), \end{aligned}\right\} \tag{35}$$

where

$$\sum_i f_i'^2 = 0, \qquad \sum_i \phi_i'^2 = 0, \tag{36}$$

the primes indicating differentiation with respect to the argument.

* It should be remarked that this result may be extended without change to the case of a minimal two-spread in n-space. Cf. Levi, *loc. cit.*, pp. 90–92.

For a minimal surface referred to its minimal lines and defined by (35), we have

$$E = G = 0, \qquad F = \Sigma f_i' \phi', \tag{37}$$

and from (28) and (29) it follows that

$$D_x' = D_y' = D_z' = D_t' = 0, \qquad e = g = 0. \tag{38}$$

From this last result and the form of equations (27) follow the theorems:

The asymptotes of the characteristic at any point on a minimal surface correspond to the directions of the minimal curves on the surface through the point.

The characteristic at each point of a minimal surface is an ellipse or a circle. (Cf. § 6.)

When the conditions (37) and (38) are satisfied, the equations (27) reduce to the form

$$X - x = \frac{F}{2f}\left(D_x'' \frac{dv}{du} - D_x \frac{du}{dv}\right). \tag{39}$$

In like manner from (28) we have

$$\left.\begin{aligned} D_x^2 &= F(2x_{10}x_{01} - F)\Sigma x_{20}^2 + F^2 x_{20}^2 + x_{10}^2\left(\frac{\partial F}{\partial u}\right)^2 - 2F\frac{\partial F}{\partial u}x_{10}x_{20}, \\ D_x D_x'' &= F(2x_{10}x_{01} - F)\Sigma x_{20}x_{02} + F^2 x_{20}x_{02} \\ &\quad + \frac{\partial F}{\partial u}\frac{\partial F}{\partial v}(F - x_{10}x_{01}) - F\frac{\partial F}{\partial u}x_{10}x_{02} - F\frac{\partial F}{\partial v}x_{01}x_{20}, \\ D_x''^2 &= F(2x_{10}x_{01} - F)\Sigma x_{02}^2 + F^2 x_{02}^2 + x_{01}^2\left(\frac{\partial F}{\partial v}\right)^2 - 2F\frac{\partial F}{\partial v}x_{01}x_{02}. \end{aligned}\right\} \tag{40}$$

From these and similar expressions for $D_y^2, \ldots, D_t''^2$, we obtain

$$\left.\begin{aligned} \Sigma D_x^2 = -F^2\Sigma x_{20}^2, \qquad \Sigma D_x''^2 = -F^2\Sigma x_{02}^2, \\ \Sigma D_x D_x'' = F\left(\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - F\Sigma x_{20}x_{02}\right). \end{aligned}\right\} \tag{41}$$

§ 4. *Normal Parameters and Functions.*

From (35) and (36) and the results of § 1 it follows that any minimal surface, whose minimal curves do not lie in planes of the type (20), may be defined by equations of the form:

$$\left.\begin{aligned} x &= \tfrac{1}{2}[\phi - u\phi' + \psi' + \phi_0 - u_0\phi_0' + \psi_0'], \\ iy &= \tfrac{1}{2}[\phi - u\phi' - \psi' - \phi_0 + u_0\phi_0' + \psi_0'], \\ z &= \tfrac{1}{2}[\psi - u\psi' - \phi' + \psi_0 - u_0\psi_0' - \phi_0'], \\ it &= -\tfrac{1}{2}[\psi - u\psi' + \phi' - \psi_0 + u_0\psi_0' - \phi_0'], \end{aligned}\right\} \tag{42}$$

where the parameters u and u_0 and the functions ϕ, ϕ_0, ψ, ψ_0 are determined by equations similar to (2) and (2').

It should be observed that the functions ϕ, ϕ, ψ, ψ_0 are determined only to within additive integral functions of the first order, provided that translations of the surface are disregarded. In fact, if these functions be replaced in (42) by the respective functions

$$\phi + ua_1 + a_2, \quad \phi_0 + u_0 b_1 + b_2, \quad \psi + uc_1 + c_2, \quad \psi_0 + u_0 d_1 + d_2,$$

it is found that the right-hand members of the above equations are increased by the respective constants

$$\begin{gathered} \tfrac{1}{2}(a_2 + c_1 + b_2 + d_1), \quad \tfrac{1}{2}(a_2 - c_1 - b_2 + d_1), \\ \tfrac{1}{2}(c_2 - a_1 + d_2 - b_1), \quad -\tfrac{1}{2}(c_2 + a_1 - d_2 + b_1). \end{gathered}$$

Conversely, by such a change of functions any translation can be effected.

In terms of these functions the coefficients E, F, G have the form

$$E = G = 0, \qquad F = \tfrac{1}{2}(1 + uu_0)(\phi''\phi_0'' + \psi''\psi_0''). \tag{43}$$

Moreover, the expressions (41) are reducible to

$$\left.\begin{aligned} &\Sigma D_x^2 = F^2 U, \qquad \Sigma D_x''^2 = F^2 U_0, \\ &\Sigma D_x D_x'' = -F\left[\frac{(1+uu_0)^2}{4} UU_0 + \frac{F^2}{(1+uu_0)^2}\right], \end{aligned}\right\} \tag{44}$$

where, for the sake of brevity, we have put

$$U = \phi''\psi''' - \phi'''\psi'', \qquad U_0 = \phi_0''\psi_0''' - \phi_0'''\psi_0''. \tag{45}$$

From these results it follows that the equation (30) of the lines of curvature on a minimal surface defined by equations (42) reduces to the simple form

$$U du^4 - U_0 du_0^4 = 0. \tag{46}$$

If we put

$$\int \sqrt[4]{U} du = \bar{u} + i\bar{v}, \qquad \int \sqrt[4]{U_0} du_0 = \bar{u} - i\bar{v}, \tag{47}$$

it follows from (46) that the lines of curvature are defined by

$$\left.\begin{aligned} &\bar{u} = \text{const.}, \qquad \bar{v} = \text{const.} \\ &\bar{u} + \bar{v} = \text{const.}, \quad \bar{u} - \bar{v} = \text{const.} \end{aligned}\right\} \tag{48}$$

Hence we have the theorem:

When equations of a real minimal surface are written in the form (42), *the determination of the line of curvature reduces to the quadrature*

$$\int \sqrt[4]{\phi''\psi''' - \phi'''\psi''}\, du. \tag{49}$$

If K denotes the Gaussian curvature* of the quadratic form which is the

* Cf. E., p. 155. A reference of this kind is to the author's Differential Geometry.

square of the linear element of the surface, when the coefficients have the form (43), then

$$K = -\frac{1}{F^3}\left[\frac{F^2}{(1+uu_0)^2} + \frac{(1+uu_0)^2}{4} UU_0\right]. \tag{50}$$

The formula (31) gives the radius of first curvature of the curve in which the surface is cut by the normal hyperplane whose direction is determined by the corresponding value of dv/du.* When the values from (43) and (44) are substituted in this expression, we have

$$\frac{1}{\rho^2} = \frac{1}{2F^3}\left[\frac{(1+uu_0)^2}{4} UU_0 + \frac{F^2}{(1+uu_0)^2}\right] - \frac{1}{4F}\left[U\left(\frac{du}{du_0}\right)^2 + U_0\left(\frac{du_0}{du}\right)^2\right]. \tag{51}$$

One shows without difficulty that the direction which makes the angle $\pi/4$ with the direction defined by

$$\frac{du_0}{du} = M(u, u_0)$$

is given by

$$\frac{du_0}{du} = iM(u, u_0).$$

Hence if ρ' denotes the radius of normal curvature in a direction making the angle $\pi/4$ with a direction whose radius of normal curvature is ρ, we have†

$$\frac{1}{\rho^2} + \frac{1}{\rho'^2} = -K. \tag{52}$$

A particular case of this is afforded by the normal curvatures in the directions of two lines of curvature inclined at the angle $\pi/4$ to one another.

One finds from (51) that the principal radii of normal curvature are given by

$$\rho^2 = \frac{2F^3}{\left[\frac{1+uu_0}{2}\sqrt{UU_0} - \frac{\varepsilon F}{1+uu_0}\right]^2}, \tag{53}$$

where $\varepsilon = \pm 1$ and the corresponding directions are given by

$$\sqrt{U}du^2 - \varepsilon\sqrt{U_0}du_0^2 = 0.$$

The necessary and sufficient condition that the lines of curvature be indeterminate is that $U = U_0 = 0$. In this case ρ is the same for all directions, and

* Levi, *loc. cit.*, p. 58.

† Cf. Hovestadt, Programm des Münster'schen Realgymnasiums (1880); also Kommerell, *loc. cit.*, p. 33.

it may be seen also from (39) that the characteristic is a circle with the point (u, u_0) of the surface for center, and only in this case. Hence we have the theorem:

A necessary and sufficient condition that the characteristic be a circle is that

$$\phi''\psi''' - \phi'''\psi'' = 0, \qquad \phi_0''\psi_0''' - \phi_0'''\psi_0'' = 0; \tag{53'}$$

in this case the center is at the point (u, u_0) of the surface, so that the locus of the centers of normal curvature for the point is the same circle, and the lines of curvature are undetermined.

§ 5. *Change of Normal Parameters. Rotations.*

The particular value of the form of equations (42) lies in the fact that if u_0 is the conjugate-imaginary of u and the functions ϕ_0 and ψ_0 are conjugate to ϕ and ψ respectively, the surface is real. In view of results of § 1 it is evident that the equations of minimal curves on the surface, and consequently the equations of the surface, may be given other similar forms. For example, the equations of the curve of parameter u_0 may be expressed in terms of a different parameter u_1 such that its equations shall be of the same form as the curves of parameter u. The conditions necessary and sufficient to this end are

$$u_1\phi_1'' - \psi_1'' = (u_0\phi_0'' - \psi_0'')\frac{du_0}{du_1}, \qquad u_1\phi_1'' + \psi_1'' = -(u_0\phi_0'' + \psi_0'')\frac{du_0}{du_1},$$
$$u_1\psi_1'' + \phi_1'' = (u_0\psi_0'' + \phi_0'')\frac{du_0}{du_1}, \qquad u_1\psi_1'' - \phi_1'' = -(u_0\psi_0'' - \phi_0'')\frac{du_0}{du_1},$$

which may be shown to be equivalent to

$$u_1 = -\frac{1}{u_0}, \qquad \phi_1 = \frac{\psi_0 - c_1}{u_0} + c_2, \qquad \psi_1 = \frac{c_3 - \phi_0}{u_0} + c_4, \tag{54}$$

where c_1, c_2, c_3, c_4 denote constants. Evidently these are necessary and sufficient conditions that the two generating curves shall be superposable by a translation. Furthermore, if we put the expressions (54) for u_1, ϕ_1, ψ_1 in equations (7) and (8), or for u_2, ϕ_2, ψ_2 in (9) and (10), and solve for u_0, ϕ_0, ψ_0, we find the necessary and sufficient conditions which these latter quantities must satisfy in order that the two generating curves be congruent. These conditions are readily found and will not be written out here.

The results of § 1 dealing with the effect upon the normal parameter and the normal functions of rotation of the coordinate axes, or what is the same thing a

rotation of the curve itself, may be extended to minimal surfaces. In fact it can be shown that the minimal surface whose normal parameters and functions are in the following relation to those of the surface (42),

$$\left.\begin{array}{llll} u_1 = u, & u_{10} = u_0, & \phi_1 = \dfrac{a_1\phi + a_2\psi}{A}, & \psi_{10} = \dfrac{a_4\phi_0 - a_3\psi_0}{A}, \\ \psi_1 = \dfrac{a_3\phi + a_4\psi}{A}, & & \psi_{10} = \dfrac{-a_2\phi_0 + a_1\psi_0}{A}, & A = a_1a_4 - a_2a_3, \end{array}\right\} \quad (55)$$

may be obtained from the surface (42) by the rotation defined by equations (13) when $e = 1$.

In like manner the surface for which

$$\left.\begin{array}{lll} u_1 = \dfrac{a_1u + a_2}{a_3u + a_4}, & u_{10} = \dfrac{a_4u_0 - a_3}{-a_2u_0 + a_1}, & A = a_1a_4 - a_2a_3, \\ \phi_1 = \dfrac{A\phi}{a_3u + a_4}, & \phi_{10} = \dfrac{A\phi_0}{-a_2u_0 + a_1}, & \\ \psi_1 = \dfrac{A\psi}{a_3u + a_4}, & \psi_{10} = \dfrac{A\psi_0}{-a_2u_0 + a_1} & \end{array}\right\} \quad (56)$$

differs from the surface (42) by the rotation defined by the equations (13) when $e = -1$.

The second of the foregoing results may also be stated as follows:

If there exists between two minimal surfaces a relation such that their normal parameters are linear fractional functions of one another, as (56), *either of the surfaces may be so rotated that the normal parameters become equal; in this case the new and old functions* ϕ *and* ψ *of the rotated surface are in the relations* (56).

§ 6. *Surfaces of Riemann.*

Kommerell* has given the name Riemann surface to the two-dimensional spread in plane four-space, defined by

$$z + it = f(x + iy), \tag{57}$$

where f is a function of the complex variable $x + iy$ analytic in a determined

* *Loc. cit.*, p. 548.

domain of this variable. It may be shown* that the most general Riemann surface is defined by

$$x = x, \qquad y = y, \qquad z = u(x, y), \qquad t = v(x, y),$$

where u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Kommerell† has shown furthermore that all of these surfaces are minimal surfaces, as defined in § 3.

We shall now determine in what manner these surfaces are characterized by the normal functions ϕ and ψ. To this end we observe that the Jacobian of the functions $x + iy$ and $z + it$ as given by (42) is

$$(1 + uu_0)\phi''\psi_0''.$$

Hence if we disregard translations, we have the theorem:

The necessary and sufficient condition that the minimal surface defined by (42) *be real and satisfy the condition* (57) *is that*

$$\phi = \phi_0 = 0 \quad \text{or} \quad \psi = \psi_0 = 0.$$

If $\phi = \phi_0 = 0$, equations (42) become

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\psi' + \psi_0'), & y &= \frac{i}{2}(\psi' - \psi_0'), \\ z &= \tfrac{1}{2}(\psi - u\psi' + \psi_0 - u_0\psi_0'), & t &= \frac{i}{2}(\psi - u\psi' - \psi_0 + u_0\psi_0'). \end{aligned}\right\} \tag{58}$$

From (58) it is evident that the curves $u_0 =$ const. lie in planes parallel to the plane

$$x + iy = 0, \qquad z + it = 0,$$

and the curves $u =$ const. in planes parallel to the plane

$$x - iy = 0, \qquad z - it = 0.$$

In § 5 we have seen the effect of a rotation of axes upon the equations of a minimal surface. Comparing equations (55) and (56) with the preceding results of this section, we have the theorem:

The necessary and sufficient condition that a minimal surface defined by (42)

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* *Loc. cit.*, p. 571. † *Loc. cit.*, p. 586.

be a real Riemann surface is that the functions ϕ, ϕ_0, ψ, ψ_0 *satisfy the conditions*

$$a\phi + b\psi = 0, \qquad a_0\phi_0 + b_0\psi_0 = 0, \tag{59}$$

where a, b *are constants, and* a_0, b_0 *are their respective conjugate imaginaries.*

From this result and (18) it follows that

The minimal surfaces of Riemann are characterized by the fact that their minimal curves lie in planes whose equations are of the form

$$a(z+it) - b(x-iy) = c, \qquad a(x+iy) + b(z-it) = d,$$

and

$$a_0(z-it) - b_0(x+iy) = c_0, \qquad a_0(x-iy) + b_0(z+it) = d_0,$$

where c, d *are constants and* c_0, d_0 *are their conjugate imaginaries.*

Since equations (59) are equivalent to (53′), we have the following theorem:

*A necessary and sufficient condition that the characteristic at each point of a surface be a circle is that the surface be of the Riemann type; the center of the circle is at the corresponding point of the surface.**

§ 7. *Algebraic Minimal Surfaces.*

If we put

$$\left.\begin{aligned} J_{12} &= \frac{\partial(x,y)}{\partial(u,u_0)}, & J_{13} &= \frac{\partial(x,z)}{\partial(u,u_0)}, & J_{14} &= \frac{\partial(x,t)}{\partial(u,u_0)}, \\ J_{23} &= \frac{\partial(y,z)}{\partial(u,u_0)}, & J_{24} &= \frac{\partial(y,t)}{\partial(u,u_0)}, & J_{34} &= \frac{\partial(z,t)}{\partial(u,u_0)}, \end{aligned}\right\} \tag{60}$$

we have

$$I^2 = J_{12}^2 + J_{13}^2 + \dots + J_{34}^2 = \Sigma x_{10}^2 \,.\, \Sigma x_{01}^2 - (\Sigma x_{10} x_{01})^2.$$

If we put further

$$P_{ij} = -P_{ji} = \frac{J_{ij}}{I} \quad \left(\begin{matrix} i = 1, \dots, 4, \\ j = 1, \dots, 4, \end{matrix}\ i \neq j\right),$$

we find that

$$\left.\begin{aligned} P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2 &= 1, \\ P_{12}P_{34} + P_{14}P_{23} - P_{13}P_{24} &= 0. \end{aligned}\right\} \tag{61}$$

We may call the functions P_{ij} the *direction-cosines of the tangent plane.*

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* Kommerell, *loc. cit.*, p. 580, shows that each surface of Riemann possesses this property, but not that it is characteristic.

When the expressions (2) are substituted in the above equations and we take

$$I = -iF = -\frac{i}{2}(1 + u u_0)\Phi_1,$$

where

$$\Phi_1 = \phi''\phi_0'' + \psi''\psi_0'', \quad \Phi_2 = \phi''\psi_0'' + \phi_0''\psi'', \quad \Phi_3 = \phi''\psi_0'' - \phi_0''\psi'',$$

we find

$$\left.\begin{aligned}
P_{12} &= \frac{\psi''\psi_0'' - u u_0 \phi''\phi_0''}{(1 + u u_0)\Phi_1}, & P_{13} &= \frac{i}{2}\left[\frac{\Phi_3}{\Phi_1} + \frac{u - u_0}{u u_0 + 1}\right], \\
P_{14} &= \frac{1}{2}\left[\frac{\Phi_2}{\Phi_1} - \frac{u + u_0}{1 + u u_0}\right], & P_{23} &= \frac{1}{2}\left[\frac{\Phi_2}{\Phi_1} + \frac{u + u_0}{1 + u u_0}\right], \\
P_{24} &= -\frac{i}{2}\left[\frac{\Phi_3}{\Phi_1} + \frac{u_0 - u}{1 + u u_0}\right], & P_{34} &= \frac{u u_0 \psi''\psi_0'' - \phi''\phi_0''}{(1 + u u_0)\Phi_1}.
\end{aligned}\right\} \quad (62)$$

From these equations we obtain the following:

$$\frac{P_{23} - P_{14}}{1 + P_{12} - P_{34}} = \frac{u + u_0}{2}, \quad \frac{P_{13} + P_{24}}{1 + P_{12} - P_{34}} = \frac{i(u - u_0)}{2}. \quad (63)$$

We are now in a position to establish the theorem:

The necessary and sufficient condition that the minimal surface defined by (42) *be a real algebraic surface is that the functions* ϕ, ϕ_0, ψ, ψ_0 *be algebraic.*

From the form of equations (42) the sufficiency of this condition is evident. In order to establish its necessity, we make use of the following theorem of Weierstrass:*

Given a function $\Phi(\xi + i\eta)$ *and let* $\Psi(\xi, \eta)$ *denote the real part of* Φ; *if in a certain domain an algebraic relation exists between* Ψ, ξ *and* η, *then* Φ *is an algebraic function of* $\xi + i\eta$ *for that domain.*

In applying this theorem, we observe that, since

$$\frac{\partial(x, y)}{\partial(u', u_0')} = \frac{\partial(x, y)}{\partial(u, u_0)} \cdot \frac{\partial(u, u_0)}{\partial(u', u_0')},$$

the direction-cosines are absolute invariants under a change of parameters. Hence, equations (63) give the relation between any set of parameters and the particular ones u, u_0 which lead to the form (42).

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* *Monatsberichte der Berliner Akademie* (1867), pp. 511–518; also E., p. 261.

An algebraic minimal surface may be defined by two algebraic equations of the form

$$z = \psi_1(x, y), \qquad t = \psi_2(x, y). \tag{64}$$

If x, y be taken for parameters in finding the functions P_{ij}, and if the real and imaginary parts of u are given by

$$u = \xi + i\eta, \qquad u_0 = \xi - i\eta,$$

then equations (63) are expressible in the form

$$\phi_1(x, y) = \xi, \qquad \phi_2(x, y) = \eta,$$

where ϕ_1 and ϕ_2 are algebraic. Eliminating y and x from these two equations, we have respectively algebraic equations of the form

$$F_1(x, \xi, \eta) = 0, \qquad F_2(y, \xi, \eta) = 0.$$

Moreover, by means of (64) we get two other algebraic equations

$$F_3(z, \xi, \eta) = 0, \qquad F_4(t, \xi, \eta) = 0.$$

Consequently (cf. 42) the real part of each of the functions

$$\left.\begin{aligned} f_1 &= \phi - u\phi' + \psi', & f_2 &= \phi - u\phi' - \psi', \\ f_3 &= \psi - u\psi' - \phi', & f_4 &= \psi - u\psi' + \phi' \end{aligned}\right\} \tag{65}$$

is an algebraic function of ξ and η; hence, by the above Weierstrass theorem the functions f_1, f_2, f_3, f_4 defined by (65) are algebraic functions of u. But

$$\phi = \frac{1}{2}(f_1 + f_2) - \frac{u}{2}(f_3 - f_4),$$

$$\psi = \frac{1}{2}(f_3 + f_4) + \frac{u}{2}(f_1 - f_2).$$

Therefore, the functions ϕ and ψ are algebraic functions of u, and in like manner ϕ_0 and ψ_0 are algebraic functions of u_0.

§ 8. *Associate Minimal Surfaces. Formulas of Schwarz.*

Just as in the case of ordinary minimal surfaces in three-space, if a minimal surface in plane four-space is defined by (35), the equations of the form

$$x_\alpha = e^{i\alpha} f_1 + e^{-i\alpha}\phi_1, \ \cdots, \tag{66}$$

where α denotes a constant, define a family of applicable minimal surfaces which

we say are *associate* to (35) and to one another. We denote by S_α the surface whose equations are (66) for a definite value of α.

If $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ denote the coordinates of the *adjoint surface* defined by (66), when $\alpha = \pi/2$, the equations (66) may be written

$$x_\alpha = \cos\alpha\, x + \sin\alpha\, \bar{x}, \; \ldots\ldots \tag{67}$$

Hence, the applicability of a family of associate minimal surfaces may be expressed in the form:

*A minimal surface admits of a continuous deformation into a series of minimal surfaces, and in the deformation each point of the surface describes an ellipse whose plane passes through a fixed point which is the center of the ellipse.**

From equations (35) and (66) we find that *tangents to corresponding curves on S and S_α make the angle α with one another.*†

Referring to equations (42) we see that the surfaces associate to S are determined by the functions

$$\phi e^{i\alpha}, \quad \phi_0 e^{-i\alpha}, \quad \psi e^{i\alpha}, \quad \psi_0 e^{-i\alpha}. \tag{68}$$

From this result and the expressions (62) for the direction-cosines of the tangent plane to S it follows that

The tangent planes at corresponding points of a family of associate minimal surfaces are parallel.

From (29) it follows that the function f has the same value at corresponding points on S and S_α. Moreover, from (28) we have that each of the functions D for S_α is equal to the product of $e^{i\alpha}$ and the corresponding D for S, and each D'' for S_α, the product of $e^{-i\alpha}$ and the corresponding D'' for S.

We consider now the point on the characteristic of S, at the point (u, u_0), corresponding to a given direction

$$\frac{du_0}{du} = M(u, u_0). \tag{69}$$

If $X_\alpha, Y_\alpha, Z_\alpha, T_\alpha$ denote the coordinates of the point on the characteristic for S_α at (u, u_0) corresponding to the direction

$$\frac{du_0}{du} = M(u, u_0)e^{i\alpha}, \tag{70}$$

* Cf. E., p. 264. † Cf. Levi, *loc. cit.*, p. 94.

we have from equations (39) and similar ones for S_a

$$X_a - x_a = X - x, \qquad Y_a - y_a = Y - y, \qquad \cdots\cdots \tag{71}$$

Since the tangent planes to S and S_a at (u, u_0) are parallel, so also are the normal planes. From this result and equations (71) we have the theorem:

The characteristics at corresponding points of a family of associate surfaces are equal ellipses, similarly placed in parallel planes.

We shall say that two points on the characteristics which are in the relation (71) are *congruent*, since they can be made to coincide by a translation of either surface such that corresponding points on the two surfaces coincide also. One finds without difficulty that the curves (70) on S make the angle $\alpha/2$ with the curves (69). Hence, we have the theorem:

Let P and P_a be congruent points on corresponding characteristics of S and an associate S_a; to the direction on S_a determining P_a corresponds on S a direction making the angle $\alpha/2$ with the direction determining P.

From the definition of lines of curvature of a surface, we have as a corollary of the preceding theorem:

*The curves on S corresponding to lines of curvature on S_a make the constant angle $\alpha/2$ with the lines of curvature on S.**

This follows likewise from (46) and the equations of the lines of curvature on S_a, namely

$$e^{2i\alpha} U du^4 - e^{-2i\alpha} U_0 du_0^4 = 0.$$

In fact, in terms of the functions u and v (cf. 48), the finite equations of the lines of curvature on S_a are

$$\cos\frac{\alpha}{2}\bar{u} - \sin\frac{\alpha}{2}\bar{v} = \text{const.},$$

$$\sin\frac{\alpha}{2}\bar{u} + \cos\frac{\alpha}{2}\bar{v} = \text{const.},$$

$$\cos\frac{\alpha}{2}(\bar{u} - \bar{v}) - \sin\frac{\alpha}{2}(\bar{u} + \bar{v}) = \text{const.},$$

$$\sin\frac{\alpha}{2}(\bar{u} - \bar{v}) + \cos\frac{\alpha}{2}(\bar{u} + \bar{v}) = \text{const.}$$

*Cf. E., p. 264.

The foregoing results may be obtained also by means of the formula (51) of the radius of normal curvature.

A minimal surface and its adjoint possess the property of orthogonality of linear elements, expressed by

$$dx\,d\bar{x} + dy\,d\bar{y} + dz\,d\bar{z} + dt\,d\bar{t} = 0. \tag{72}$$

Since the tangent plane to $S_{\pi/2}$ is parallel to the tangent plane of S, we have

$$\left\| \begin{array}{cccc} dx_1 & dy_1 & dz_1 & dt_1 \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} & \dfrac{\partial t}{\partial u} \\ \dfrac{\partial x}{\partial v_0} & \dfrac{\partial y}{\partial u_0} & \dfrac{\partial z}{\partial u_0} & \dfrac{\partial t}{\partial v} \end{array} \right\| = 0. \tag{73}$$

From (72), (73) and § 7 we obtain

$$\frac{dx_1}{P_{12}dy + P_{13}dz + P_{14}dt} = \frac{dy_1}{P_{23}dz + P_{24}dt + P_{21}dx}$$
$$= \frac{dz_1}{P_{34}dt + P_{31}dx + P_{32}dy} = \frac{dt_1}{P_{41}dx + P_{42}dy + P_{43}dz} = \lambda, \tag{74}$$

where λ denotes the common value. By means of (62), (42) and (67), it may be shown that $\lambda = 1$. Hence we have in (74) a set of formulas analogous to the celebrated *formulas of Schwarz* for ordinary minimal surfaces.*

As in the case of ordinary minimal surfaces, these formulas enable one to find a minimal surface passing through a given curve in space and tangent to a given plane at each point. In fact, it may be shown that if x, y, z, t as functions of a real parameter v define a curve in space, and P_{ij} as functions of v determine planes passing through corresponding points of the curve, one at each point, and if ω denotes a complex variable, and $R(\theta)$ the real part of an analytic function θ of ω, then the equations

$$x_1 = R[x(\omega) + i\int_v^\omega P_{12}dy + P_{13}dz + P_{14}dt$$

and similar expressions for y_1, z_1, t_1 define the minimal surface desired.

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* Cf. E., p. 264.

As a consequence of this result we have the theorems analogous to theorems in the theory of ordinary minimal surfaces:*

If a straight line lies in a minimal surface, the surface is symmetric with respect to this line.

If a hyperplane is normal to a surface at all of its points of intersection, the surface is symmetric with respect to the plane.

PRINCETON UNIVERSITY, *June*, 1911.

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* Cf. E., pp. 266, 267.